Two-dimensional gravity with an invariant energy scale

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Abstract

We investigate the gauging of a two-dimensional deformation of the Poincaré algebra, which accounts for the existence of an invariant energy scale. The model describes 2D dilaton gravity with torsion. We obtain explicit solutions of the field equations and discuss their physical properties.

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Investigation of quantum gravity and string theory seems to indicate the existence of a fundamental length scale of the order of the Planck length [1], that may also give rise to observable effects [2]. However, a fundamental frame-independent length (or equivalently, energy scale) cannot be introduced without modifying special relativity, since it would break the invariance of the theory under the Poincaré group [3]. Recently, it was observed by Magueijo and Smolin (MS) [4] that it is nevertheless possible to preserve the invariance under the subgroup of Lorentz transformations, assuming that its action on momentum space is non-linear. As remarked in [5], this proposal can be interpreted as a special case of a larger class of deformations of the Poincaré algebra which were introduced in [6]. The effect of these deformations would be appreciable only for energy scales of the order of the Planck energy, while for smaller scales one would recover special relativity.

An interesting problem is how to include gravity in this framework. One may hope that the modification of the short-distance behaviour of the theory induced by the existence of a minimal length could avoid the singularities which affect general relativity. Of course, the most straightforward way to introduce gravity is by gauging the deformed Poincaré algebra of Ref. [4]. This algebra can be considered as a special case of non-linear algebra. The gauge theory of non-linear algebras has been studied some time ago [7, 8], but unfortunately a suitable action for these models has been obtained only in two dimensions [7][1].

In this letter, we apply the formalism of [7] to the study of the two-dimensional version of the MS algebra. We obtain a model of 2D gravity that modifies those based on the Poincaré algebra [10]. An interesting consequence of the breaking of the Poincaré invariance is that non-trivial torsion is present in the theory. This seems to be a essential feature of models of this kind.

The non-linear deformed Poincaré algebra of [4] is given in two dimensions by the commutation relations

$$[P_a, P_b] = 0, \quad [J, P_0] = \left(1 - \frac{P_0}{\kappa}\right) P_1, \quad [J, P_1] = P_0 - \frac{P_1^2}{\kappa}, \tag{1}$$

where P_a are the generators of translations and J that of boosts and $a = 0, 1$. Tangent space indices are lowered and raised by the tensor $h_{ab} = \mathrm{diag}(-1, 1)$. We also make use of the antisymmetric tensor ϵ_{ab}, with $\epsilon_{01} = 1$. The deformation parameter κ has the dimension of a mass and can be identified with

[1] An equivalent formalism was introduced also in [9].

the inverse of the Planck length. The algebra admits a Casimir invariant

$$M^2 = \frac{P_1^2 - P_0^2}{\left(1 - \frac{P_0}{\kappa}\right)^2}. \tag{2}$$

In the following it will be useful to denote the generators of the algebra as T_A, where $A = 0, 1, 2$ and $T_a = P_a$, $T_2 = J$.

In order to construct a gauge theory for this algebra, we adopt the formalism of Ikeda [7]. Given an algebra with commutation relations $[T_A, T_B] = W_{AB}(T)$, one introduces gauge fields A^A and a coadjoint multiplet of scalar fields η_A, which under infinitesimal transformations of parameter ξ^A transform as

$$\begin{aligned} \delta A^A &= \mathrm{d}\xi^A + U^A_{BC}(\eta) A^B \xi^C, \\ \delta\eta_A &= -W_{AB}(\eta)\xi^B, \end{aligned} \tag{3}$$

where U^A_{BC} and W_{AB} are functions of the fields η which satisfy

$$U^A_{BC} = \frac{\partial W_{BC}}{\partial \eta_A}. \tag{4}$$

One can then define the covariant derivative of the scalar multiplet

$$\mathrm{D}\eta_A = \mathrm{d}\eta_A + W_{AB} A^B, \tag{5}$$

and the curvature of the gauge fields

$$F^A = \mathrm{d}A^A + U^A_{BC} A^B \wedge A^C. \tag{6}$$

In two dimensions, a gauge invariant lagrangian density can be defined as [7]

$$L = \eta_A F^A + (W_{BC} - \eta_A U^A_{BC}) A^B \wedge A^C, \tag{7}$$

and generates the field equations

$$\mathrm{D}\eta^A = 0, \qquad F^A = 0. \tag{8}$$

In our case, the functions W_{AB} can be deduced from the algebra (1). Now one can build a theory of gravity as in [10], identifying A^a with the zweibeins e^a and A^2 with the spin connection ω. It follows that $F^2 = R$

and $F^a = T^a - \frac{\omega}{\kappa} \wedge (\eta_1 e^a + \eta_b e^b \delta^a_1)$, where $R = \mathrm{d}\omega$ is the curvature and $T^a = \mathrm{d}e^a + \epsilon^a{}_b\, \omega \wedge e^b$ the torsion, and the lagrangian (7) takes the form

$$L = \eta_a T^a + \eta_2 R + \frac{\eta_1}{\kappa} \eta_a\, \omega \wedge e^a. \tag{9}$$

Clearly, the last term in (9) breaks the Poincaré invariance.

The field equations (8) read explicitly

$$\begin{aligned}
&\mathrm{d}\eta_0 - \omega \left(\eta_1 - \tfrac{\eta_1\eta_0}{\kappa}\right), \qquad && \mathrm{d}e^0 + \omega \wedge \left(-\frac{\eta_1}{\kappa} e^0 + e^1\right) = 0, \\
&\mathrm{d}\eta_1 - \omega \left(\eta_0 - \tfrac{\eta_1^2}{\kappa}\right), && \mathrm{d}e^1 + \omega \wedge \left[\left(1 - \frac{\eta_0}{\kappa}\right) e^0 - \frac{2\eta_1}{\kappa}\right] e^1 = 0, \\
&\mathrm{d}\eta_2 + \left(\eta_1 - \tfrac{\eta_1\eta_0}{\kappa}\right) e^0 + \left(\eta_0 - \tfrac{\eta_1^2}{\kappa}\right) e^1, && \mathrm{d}\omega = 0.
\end{aligned} \tag{10}$$

They can be solved generalizing a method introduced by Solodukhin [11] for a different 2D gravity model. First define new fields $\bar\eta_a = \left(1 - \frac{\eta_0}{\kappa}\right)^{-1} \eta_a$, which satisfy the relations $\mathrm{d}\bar\eta_a = \epsilon_a{}^b \bar\eta_b$ and $\mathrm{d}(\bar\eta_a \bar\eta^a) = 0$. Hence $\bar\eta^2$ is a constant of the motion, $\bar\eta^2 = a^2$, say. We can then define a variable θ such that $\bar\eta_0 = a \sinh\theta$, $\bar\eta_1 = a\cosh\theta$, and therefore $\bar\eta_a \epsilon^{ab} \mathrm{d}\bar\eta_b = a^2 \mathrm{d}\theta$. But from the field equations (10), $\bar\eta_a \epsilon^{ab} \mathrm{d}\bar\eta_b = \omega \bar\eta_a \bar\eta^a = a^2 \omega$, and hence $\omega = \mathrm{d}\theta$, in accordance with the field equation $\mathrm{d}\omega = 0$. Moreover, from (10), $\mathrm{d}\Big[\left(1 - \frac{\eta_0}{\kappa}\right) \eta_a e^a\Big] = 0$, and one can therefore define a new variable ϕ, such that $\eta_a e^a = \left(1 - \frac{\eta_0}{\kappa}\right)^{-1} \mathrm{d}\phi$. Finally, combining the last equation with the field equation $\epsilon^a{}_b \eta_a e^b = \mathrm{d}\eta_2 - \frac{\eta_1}{\kappa} \eta_a e^a$, one can solve for e^0 and e^1.

One is still free to choose a gauge. The most interesting choices are $\theta = 0$ or $\mathrm{d}\theta = \Delta \mathrm{d}\phi$, where $\Delta = \left(1 - \frac{\eta_0}{\kappa}\right)^{-1} = 1 + \frac{a}{\kappa} \sinh\theta$. The first choice leads to flat space with vanishing torsion. In the second case,

$$e^0 = -\frac{\Delta}{a} (\cosh\theta \mathrm{d}\psi + \sinh\theta \mathrm{d}\theta), \qquad e^1 = \frac{\Delta}{a} (\sinh\theta \mathrm{d}\psi + \cosh\theta \mathrm{d}\theta), \tag{11}$$

where we have defined a new coordinate $\psi = \eta_2 - \log\Delta$.

The corresponding metric is given by

$$ds^2 = \left(1 + \frac{a}{\kappa} \sinh a\theta\right) (-\mathrm{d}\psi^2 + \mathrm{d}\theta^2), \tag{12}$$

and the components of the torsion are

$$T^0 = -\frac{a^2}{\kappa} \cosh^2 a\theta\ \mathrm{d}\theta \wedge \mathrm{d}\psi, \qquad T^1 = \frac{a^2}{\kappa} \cosh a\theta \sinh a\theta\ \mathrm{d}\theta \wedge \mathrm{d}\psi, \tag{13}$$

where we have rescaled the coordinates by $1/a$ in order to give them the physical dimension of a length. This solution is of black hole type; in fact, the metric vanishes at $a\theta_0 = \text{arcsinh}(-\kappa/a)$. However, while the curvature is zero everywhere, the invariants built with the torsion diverge at $\theta = 0$, and hence the solution is singular there. It is also worth noticing that the solution deviates from flat space for "mass" a much greater than the Planck energy κ.

A regular solution of the field equations can be obtained if $\bar{\eta}^2 < 0$, $\bar{\eta}^2 = -a^2$. In this case the solution has the form

$$\mathrm{d}s^2 = \left(1 + \frac{a}{\kappa} \cosh a\theta\right) (-\mathrm{d}\theta^2 + \mathrm{d}\psi^2), \tag{14}$$

with

$$T^0 = -\frac{a^2}{\kappa} \sinh^2 a\theta \ \mathrm{d}\theta \wedge \mathrm{d}\psi, \qquad T^1 = \frac{a^2}{\kappa} \cosh a\theta \sinh a\theta \ \mathrm{d}\theta \wedge \mathrm{d}\psi. \tag{15}$$

This solution presents no singularities, but the coordinate θ is now timelike. The solution is therefore of cosmological type and describes a universe which evolves from an infinite size at $t = -\infty$ to a minimum size and then expands forever.

We have shown that it is possible to construct a theory of gravity in two dimensions based on the MS algebra. This models admits cosmological solutions, but no black hole solution with regular horizon. Of course, this is the simplest model one can imagine (its Poincaré-invariant limit possesses only flat solutions) and one may consider models based on different deformations of 2D Poincaré or de Sitter algebras, which may show more attractive features. Also, it would be interesting to introduce matter couplings in order to obtain more physical insight on the properties of the theory. Anyway, it seems that the introduction of the new invariant parameter κ affects the global properties of the solutions, rather than their short-distance behavior. Perhaps the introduction of non-commuting spacetime coordinates is necessary for this purpose [6].

The most interesting development of our results would be of course their extension to higher dimensions. This would require the definition of an action suitable for non-linear gauge theories in $D > 2$, which is not known at present. An important point however is that all models of this kind imply the presence of non-trivial torsion. One can conjecture that this is the only effect that may cause a breaking of the equivalence principle in this framework.

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